Math 33A:

HW 2 Due Mon., 7/15, 11:59 pm

<u>OH</u>: Today, 3-4 pm

Mon, 1-2 pm

Working Example:

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

1. Find the QR factorization of A

2. Find the least-squares solution to $A\vec{x} = \vec{b}$

   (Bonus: Can we find it using the QR factorization?)

Orthonormal basis:

subspace V, a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthonormal if all the vectors are pairwise orthogonal $(\vec{v}_i \cdot \vec{v}_j = 0,\ i \neq j)$ & $\|\vec{v}_i\| = 1$ $(\vec{v}_i \cdot \vec{v}_i = 1)$

Gram-Schmidt Algorithm:

Basis $\rightsquigarrow$ Orthonormal basis

We only consider QR for matrices A w/ lin. ind. cols.

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array} \end{array}$$

, find QR factorization

$$A = QR$$

↳ Has orthonormal cols.

cols. of Q give an orthonormal basis for $\operatorname{im}(A)$

$$\vec{v}_1, \vec{v}_2 \rightsquigarrow \vec{v}_1^{\perp}, \vec{v}_2^{\perp} \rightsquigarrow \vec{u}_1, \vec{u}_2$$

orthogonal

orthonormal

$$\vec{v}_1^{\perp} = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \operatorname{proj}_{\vec{v}_1^{\perp}} \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \rightarrow \sqrt{\frac{1}{4} + \frac{1}{4} + 1}$$

$$= \sqrt{\frac{3}{2}}$$

$$= \sqrt{3}/\sqrt{2}$$

$$u_1 = \frac{1}{\|v_1^\perp\|} v_1^\perp = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\|v_2^\perp\|} v_2^\perp = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$\{v_1, v_2, v_3\}$

$\vec{v}_1^\perp = \vec{v}_1$

$\vec{v}_2^\perp = \vec{v}_2 - \mathrm{proj}_{\vec{v}_1^\perp} \vec{v}_2$

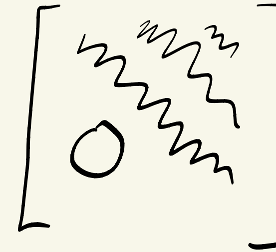
$\vec{v}_3^\perp = \vec{v}_3 - \mathrm{proj}_{v_1^\perp} \vec{v}_3 - \mathrm{proj}_{v_2^\perp} \vec{v}_3$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\underset{3\times 2}{\overset{m\times n}{A}} = \underset{3\times 2}{\overset{m\times n}{Q}}\ \underset{2\times 2}{\overset{n\times n}{R}}$$

upper triangular matrix

(square)



r_{ij} : entry of R in i^{th} row, j^{th} col.

$$r_{11} = \|\vec{v}_1\|$$

$$r_{ii} = \|\vec{v}_i^{\perp}\| \quad i > 1$$

$$r_{ij} = \begin{cases} \vec{u}_i \cdot \vec{v}_j & i < j \\ 0 & i > j \end{cases}$$

$(\vec{u}_1 \cdot \vec{v}_2)$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1/\sqrt{2}$$

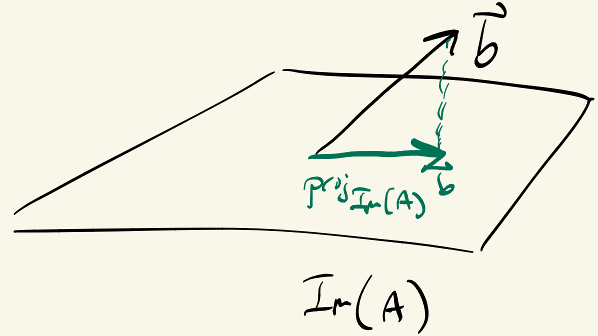
$$A = QR, \qquad Q^TQ = I$$

$$Q^TA = Q^TQR = IR = R$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{bmatrix}$$

$$\text{"} \qquad \sqrt{3}/\sqrt{2}$$

(2) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$



$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & 3 \end{array}\right]$$

Normal equation: $A^T A \vec{x} = A^T \vec{b}$

(A lin. ind cols.)

($A^T A$ is symmetric)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A^T A$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 6 \end{array}\right] /2 \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1/2 & 3 \\ 1 & 2 & 6 \end{array}\right] -(\mathrm{I}) \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1/2 & 3 \\ 0 & 3/2 & 3 \end{array}\right]$$

$$2/3 \cdot (\mathrm{II})$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1/2 & 3 \\ 0 & 1 & 2 \end{array}\right] - 1/2\,(\mathrm{II}) \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\boxed{x^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

$$A x^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A \vec{x} = A^T \vec{b} \qquad A = QR$$

$$(QR)^T (QR) \vec{x} = (QR)^T \vec{b}$$

$$R^T \underbrace{Q^T Q}_{} R \vec{x} = R^T Q^T \vec{b}$$

$$\cancel{R^T} R \vec{x} = \cancel{R^T} Q^T \vec{b}$$

$$\boxed{R\vec{x} = Q^T \vec{b}}$$

Normal Eqn

$$\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix} \vec{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{2} \\ 6/\sqrt{6} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} \sqrt{2} & 1/\sqrt{2} & 6/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 6/\sqrt{6} \end{array}\right]$$

$$\frac{\sqrt{3}}{\sqrt{2}} x_2 = \frac{6}{\sqrt{6}} = \sqrt{6}$$

$$\sqrt{3}\, x_2 = \sqrt{12} \quad \Rightarrow$$

$$x_2 = \sqrt{4} = 2$$

$$\sqrt{2}\, x_1 + \frac{x_2}{\sqrt{2}} = 6/\sqrt{2}$$

$$\sqrt{2}\, x_1 + \sqrt{2} = \frac{6}{\sqrt{2}}$$

$$\sqrt{2}\, x_1 = \frac{6}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$x_1 = \frac{4}{\sqrt{4}} = \sqrt{4} = 2$$